University of California, Berkeley
Physics 105 Fall 2000 Section 2 (*Strovink*)

SOLUTION TO PROBLEM SET 11

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Reading:
105 Notes 14.1-14.5
Hand & Finch 2.9, 9.7

1.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$\begin{aligned} y(x,0) &= A \sin \frac{3\pi x}{L} \\ \dot{y}(x,0) &= 0 \ . \end{aligned}$$

Resolve the solution into normal modes.
Solution:
A general solution to this problem can be written as:

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi x}{L}$$

where $\omega_n = n\omega_1$ and $\omega_1 = \frac{\pi}{L}\sqrt{\frac{\tau}{\mu}}$. From our initial conditions, we have:

$$\begin{aligned} y(x,0) &= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \\ &= A \sin \frac{3\pi x}{L} \end{aligned}$$

Thus, by inspection, $B_3 = A$, and all the other B_n are zero. We also have the initial condition for velocity:

$$\begin{aligned} \dot{y}(x,0) &= \sum_{n=1}^{\infty} \omega_n A_n \sin \frac{n\pi x}{L} \\ &= 0 \end{aligned}$$

Thus all the A_n are zero. So the full solution is:

$$y(x,t) = A \cos \omega_3 t \sin \frac{3\pi x}{L}$$

where $\omega_3 = \frac{3\pi}{L}\sqrt{\frac{\tau}{\mu}}$.

2.
Discuss the motion of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when (in a certain set of units) the initial conditions are

$$\begin{aligned} y(x,0) &= 4\,\frac{x(L-x)}{L^2} \\ \dot{y}(x,0) &= 0 \ . \end{aligned}$$

Find the characteristic frequencies and calculate the amplitude of the n^{th} mode.
Solution:
Again using the expansion

$$y(x,t) = \sum_{n=1}^{\infty} (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \frac{n\pi x}{L} \ ,$$

we fit the initial conditions:

$$\begin{aligned} \dot{y}(x,0) &= \sum_{n=1}^{\infty} \omega_n A_n \sin \frac{n\pi x}{L} \\ &= 0 \end{aligned}$$

Thus all the A_n are zero.

$$\begin{aligned} y(x,0) &= \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L} \\ &= \frac{4x(L-x)}{L^2} \end{aligned}$$

where

$$B_n = \frac{2}{L}\int_0^L \frac{4x(L-x)}{L^2} \sin \frac{n\pi x}{L}\, dx$$

Here are two useful integrals:

$$\int_0^L x \sin\left(\frac{n\pi x}{L}\right)\,dx = (-1)^{n+1}\frac{L^2}{n\pi}$$
$$\int_0^L x^2 \sin\left(\frac{n\pi x}{L}\right)\,dx = \begin{cases} \frac{L^3}{n\pi} - \frac{4L^3}{n^3\pi^3} & n \text{ odd} \\ \frac{-L^3}{n\pi} & n \text{ even} \end{cases}$$

So the amplitude of the nth mode is

$$B_n = \begin{cases} 32/n^3\pi^3 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

with characteristic frequencies given, as usual, by $\omega_n = \frac{n\pi}{L}\sqrt{\frac{\tau}{\mu}}$.

3.

Solve for the motion $y(x,t)$ of a continuous string (tension τ, mass per unit length μ) with fixed endpoints $y = 0$ at $x = 0$ and $x = L$, when the initial conditions are

$$y(x,0) = A \sin\frac{\pi x}{L}$$
$$\dot{y}(x,0) = V \sin\frac{5\pi x}{L} \ ,$$

where A and V are constants.

Solution:

Using the same expansion for the solution as in **1.** and **2.**, we apply the initial conditions:

$$y(x,0) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{L} = A\sin\frac{\pi x}{L} \ ,$$

from which we can see that $B_1 = A$ and all the other B_n are zero. We also have:

$$\dot{y}(x,0) = \sum_{n=1}^{\infty} \omega_n A_n \sin\frac{n\pi x}{L} = V\sin\frac{5\pi x}{L}$$

By inspection, $A_5 = \frac{V}{\omega_5}$, and all the other A_n are zero. Thus the solution is

$$y(x,t) = \frac{V}{\omega_5}\sin\omega_5 t\sin\frac{5\pi x}{L} + A\cos\omega_1 t\sin\frac{\pi x}{L} \ ,$$

where as usual $\omega_n = \frac{n\pi}{L}\sqrt{\frac{\tau}{\mu}}$.

4.

A continuous string (tension τ, mass per unit length μ) is attached to fixed supports *infinitely far away*. At $t = 0$ the string satisfies initial conditions

$$y(x,0) = 0$$
$$\frac{\partial y}{\partial t}(x,0) = \alpha\,\delta(x) \ ,$$

where $\delta(x)$ is a Dirac delta function and α is a constant that can be made arbitrarily infinitesimal, so that the string's slope remains small enough for the usual wave equation to apply. This initial condition is appropriate to the string having been struck at $(x = 0, t = 0)$ with a sharp object.

Compute $y(x,t)$ for $t > 0$.

Solution:

From Notes eqn. 14.5, we have:

$$\begin{aligned} y(x,t) &= \frac{1}{2}\big(y_0(x-ct) + y_0(x+ct)\big) \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} v_0(u)\,du \\ &= \frac{1}{2}(0+0) + \frac{1}{2c}\int_{x-ct}^{x+ct}\alpha\,\delta(u)\,du \end{aligned}$$

The integral is $\frac{\alpha}{2c}$ if the interval $(x-ct, x+ct)$ contains 0, zero otherwise. Therefore:

$$y(x,t) = \begin{cases} \frac{\alpha}{2c} & \text{if } -ct < x < ct \\ 0 & \text{otherwise} \end{cases}$$

(Here $c \equiv \sqrt{\frac{\tau}{\mu}}$.)

5.

Show that if ψ and ψ^* are taken as two *independent* field variables, the Lagrangian density

$$\mathcal{L}' = \frac{\hbar^2}{2m}\nabla\psi^*\cdot\nabla\psi + V\psi^*\psi + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

(where $\dot{}$ means $\partial/\partial t$ in this context) leads to the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\psi + V\psi = i\hbar\frac{\partial\psi}{\partial t}$$

and its complex conjugate.
Solution:
The above expression for the Lagrangian can be written as:

$$\mathcal{L}' = \frac{\hbar^2}{2m}(\partial_j\psi^*)(\partial_j\psi) + V\psi^*\psi + \frac{\hbar}{2i}(\psi^*\dot{\psi} - \psi\dot{\psi}^*)$$

where we are using the summation convention, and $\partial_j \equiv \frac{\partial}{\partial x_j}$. Our Euler-Lagrange equation looks like:

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) + \frac{d}{dx_k}\left(\frac{\partial\mathcal{L}}{\partial(\partial_k\psi)}\right) = \frac{\partial\mathcal{L}}{\partial\psi}$$

where

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\psi}}\right) = \frac{\hbar}{2i}\dot{\psi}^*$$
$$\frac{d}{dx_k}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_k\psi\right)}\right) = \frac{\hbar^2}{2m}\frac{d}{dx_k}\left(\partial_k\psi^*\right) = \frac{\hbar^2}{2m}\nabla^2\psi^*$$
$$\frac{\partial\mathcal{L}}{\partial\psi} = V\psi^* - \frac{\hbar}{2i}\dot{\psi}^*$$

Putting all these into the Euler-Lagrange formula, we get

$$\frac{\hbar}{2i}\dot{\psi}^* + \frac{\hbar^2}{2m}\nabla^2\psi^* = V\psi^* - \frac{\hbar}{2i}\dot{\psi}^*$$

Rearrange that to get

$$-\frac{\hbar^2}{2m}\nabla^2\psi^* + V\psi^* = -i\hbar\dot{\psi}^*$$

which is the complex conjugate of the usual Schrödinger equation. If you apply the Euler-Lagrange equation for ψ^*, you'll get the unconjugated Schrödinger equation.

6.
Consider a membrane stretched between *fixed* supports at $x = 0$, $x = L$, $y = 0$, and $y = L$. *Per unit area*, its kinetic and potential energies are

$$T' = \tfrac{1}{2}\sigma\left(\frac{\partial z}{\partial t}\right)^2$$
$$U' = \tfrac{1}{2}\beta\Big(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\Big) ,$$

where σ is the membrane's mass per unit area, β is a constant that is inversely proportional to its elasticity, and z is its (normal) displacement.

Apply the Euler-Lagrange equations to obtain a partial differential equation for $z(x, y, t)$. Using a trial solution

$$z(x, y, t) = X(x)\,Y(y)\,T(t) ,$$

find the angular frequencies of vibration for the *five* lowest-frequency normal modes of oscillation.
Solution:
Our Lagrangian is

$$\mathcal{L}' = \frac{1}{2}\sigma\left(\frac{\partial z}{\partial t}\right)^2 - \frac{1}{2}\beta\left(\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right)$$

Apply the Euler-Lagrange equation:

$$\frac{d}{dx}\frac{\partial\mathcal{L}'}{\partial\frac{\partial z}{\partial x}} + \frac{d}{dy}\frac{\partial\mathcal{L}'}{\partial\frac{\partial z}{\partial y}} + \frac{d}{dt}\frac{\partial\mathcal{L}'}{\partial\frac{\partial z}{\partial t}} = \frac{\partial\mathcal{L}'}{\partial z}$$
$$-\frac{d}{dx}\left(\beta\frac{\partial z}{\partial x}\right) - \frac{d}{dy}\left(\beta\frac{\partial z}{\partial y}\right) + \frac{d}{dt}\left(\sigma\frac{\partial z}{\partial t}\right) = 0$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} - \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} = 0 ,$$

where $c \equiv \sqrt{\frac{\beta}{\sigma}}$. Substituting in a solution of the form $z(z, y, t) = X(x)Y(y)T(t)$ and dividing through by z yields:

$$\frac{X''}{X} + \frac{Y''}{Y} - \frac{1}{c^2}\frac{T''}{T} = 0$$

By the usual separation of variables reasoning, each term must be separately equal to a constant. Therefore we try a solution of the form

$$X(x) \propto \sin\frac{n\pi x}{L}$$
$$Y(y) \propto \sin\frac{m\pi x}{L}$$
$$T(t) \propto e^{i\omega t}$$

where n and m are positive integers. Note that the coefficients of x are chosen to satisfy the boundary conditions $X(0) = X(L) = Y(0) =$

$Y(L) = 0$. In order to still satisfy the separated D.E., we must have

$$-\left(\frac{n\pi}{L}\right)^2 - \left(\frac{m\pi}{L}\right)^2 + \frac{\omega^2}{c^2} = 0$$
$$\omega^2 = \frac{c^2\pi^2}{L^2}(n^2 + m^2)$$

and so the frequencies of the five lowest frequency modes are given by:

$$\omega^2 = \frac{c^2\pi^2}{L^2}\begin{cases} 1^2 + 1^2 = 2 \\ 1^2 + 2^2 = 5 \\ 2^2 + 1^2 = 5 \\ 2^2 + 2^2 = 8 \\ 3^2 + 1^2 = 10 \\ 1^2 + 3^2 = 10 \\ 2^2 + 3^2 = 13 \\ 3^2 + 2^2 = 13 \end{cases}$$

7. and **8.** (double problem)
The Lagrangian density (per unit volume) for a charge density $\rho(\mathbf{r}, t)$ and current density $\mathbf{j}(\mathbf{r}, t)$ in the presence of an electromagnetic field $\mathbf{E}(\mathbf{r}, t)$, $\mathbf{B}(\mathbf{r}, t)$ is

$$\mathcal{L}' = \frac{E^2 - B^2}{8\pi} - \rho\phi + \frac{1}{c}\mathbf{j}\cdot\mathbf{A}\ .$$

The first term is the Lagrangian density corresponding to the self-energy of the free field, and the latter terms represent the interaction between fields and charges. The self-energy of the individual (point) charges is infinity in classical theory and is omitted. In the above, $\mathbf{A}$ is the *vector potential* defined by

$$\mathbf{B} = \nabla\times\mathbf{A}$$
$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}$$

(Gaussian units are used throughout this problem). If you are familiar with relativistic transformations of electromagnetic fields, you may notice that the above Lagrangian density is *Lorentz invariant*, although not manifestly so.

The homogeneous (charge and current independent) Maxwell equations follow directly from the equations relating $\mathbf{E}$ and $\mathbf{B}$ to the potentials. To complete the picture, using ϕ and the three components of $\mathbf{A}$ as four generalized (field) coordinates, apply the Euler-Lagrange equations to $\mathcal{L}'$ to obtain the two inhomogeneous Maxwell equations

$$\nabla\cdot\mathbf{E} = 4\pi\rho$$
$$\nabla\times\mathbf{B} - \frac{1}{c}\frac{\partial\mathbf{E}}{\partial t} = \frac{4\pi}{c}\mathbf{j}\ .$$

Solution:
Remember the repeated-index summation convention; we'll be using it a lot. The ith component of the electric field is $E_i = -(\partial_i\phi + \dot{A}_i/c)$. So E^2 is

$$E^2 = E_iE_i = (\partial_i\phi + \dot{A}_i)(\partial_i\phi + \dot{A}_i)\ ,$$

and B^2 is

$$B^2 = (\nabla\times\vec{A})^2 = (\nabla\times\vec{A})_i(\nabla\times\vec{A})_i$$
$$= \epsilon_{ijk}(\partial_jA_k)\epsilon_{ilm}(\partial_lA_m)\ .$$

So the Lagrangian density is

$$\mathcal{L} = \frac{1}{8\pi}\left(\partial_i\phi + \frac{1}{c}\dot{A}_i\right)\left(\partial_i\phi + \frac{1}{c}\dot{A}_i\right) - \frac{1}{8\pi}\epsilon_{ijk}\epsilon_{ilm}(\partial_jA_k)(\partial_lA_m) - \rho\phi + \frac{1}{c}j_iA_i\ .$$

The Euler-Lagrange equation for a coordinate η is

$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{\eta}}\right) + \frac{d}{dx_a}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial_a\eta\right)}\right) - \frac{\partial\mathcal{L}}{\partial\eta} = 0$$

(Note that there is an implied sum over a in the second term.) We have four coordinates: ϕ and the three components of $\vec{A}$. Let's start by setting $\eta = \phi$. Then the first term is zero, and the third term is $\partial\mathcal{L}/\partial\phi = -\rho$. To figure out the second term, note that $\partial_a\phi$ occurs in the Lagrangian density only in the first term, and only when $i = a$. So

$$\frac{\partial\mathcal{L}}{\partial\left(\partial_a\phi\right)} = \frac{1}{4\pi}\left(\partial_a\phi + \frac{1}{c}\dot{A}_a\right)$$

Putting all of this into the Euler-Lagrange equation, we get

$$\frac{1}{4\pi}\frac{d}{dx_a}\left(\partial_a\phi + \frac{1}{c}\dot{A}_a\right) + \rho = 0$$

The first term is just $-(1/4\pi)dE_a/dx_a$, which is $-\nabla \cdot \vec{E}/4\pi$, so this equation is

$$\nabla \cdot \vec{E} = 4\pi\rho$$

Now let's choose as our coordinate an arbitrary component of the vector potential: $\eta = A_b$. Then

$$\begin{aligned}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{A}_b}\right) &= \frac{1}{4\pi c}\frac{d}{dt}\left(\partial_b\phi + \frac{1}{c}\dot{A}_b\right)\\ &= -\frac{1}{4\pi c}\frac{dE_b}{dt}\end{aligned}$$

and

$$\begin{aligned}&\frac{d}{dx_a}\left(\frac{\partial \mathcal{L}}{\partial\left(\partial_a A_b\right)}\right)\\ &= -\frac{1}{8\pi}\frac{d}{dx_a}\left(\epsilon_{iab}\epsilon_{ilm}\partial_l A_m + \epsilon_{ijk}\epsilon_{iab}\partial_j A_k\right)\\ &= -\frac{1}{4\pi}\epsilon_{iab}\epsilon_{ilm}\partial_a\partial_l A_m\end{aligned}$$

and

$$\frac{\partial \mathcal{L}}{\partial A_b} = \frac{j_b}{c}$$

The second expression above requires some explanation. The term in the Lagrangian density that involves spatial derivatives of $\vec{A}$ has no fewer than five implied summations: i, j, k, l, m are all summed over. The derivative with respect to $\partial_a A_b$ gets nonzero contributions when $(j, k) = (a, b)$ and when $(l, m) = (a, b)$. Those are the two terms in the second line of the second expression above. Those two terms are equal, as you can see by relabeling the remaining dummy indices. (Specifically, relabel j, k to be l, m in the second term.)

So the Euler-Lagrange equation is

$$-\frac{1}{4\pi c}\frac{dE_b}{dt} - \frac{1}{4\pi}\epsilon_{iab}\epsilon_{ilm}\partial_a\partial_l A_m - \frac{j_b}{c} = 0$$

But

$$\begin{aligned}\epsilon_{iab}\epsilon_{ilm}\partial_a\partial_l A_m &= \epsilon_{iab}\partial_a\left(\nabla \times \vec{A}\right)_i\\ &= -\epsilon_{bai}\partial_a B_i\\ &= -\left(\nabla \times \vec{B}\right)_b\end{aligned}$$

So the Euler-Lagrange equation for A_b becomes

$$\frac{1}{c}\frac{dE_b}{dt} - \left(\nabla \times \vec{B}\right)_b + \frac{j_b}{4\pi c} = 0$$

which is just the b component of the second Maxwell equation.